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PHYSICAL INTERPRETATION OF THE PROPER ORTHOGONAL MODES USING THE SINGULAR VALUE DECOMPOSITION

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Proper orthogonal decomposition is a statistical pattern analysis technique for finding the dominant structures, called the proper orthogonal modes, in an ensemble of spatially distributed data. While the proper orthogonal modes are obtained through a statistical formulation, they can be physically interpreted in the field of structural dynamics. The purpose of this paper is thus to provide some insights into the physical interpretation of the proper orthogonal modes using the singular value decomposition

1. INTRODUCTION

Proper orthogonal decomposition (POD) is a procedure for extracting a basis for a modal decomposition from an ensemble of signals. A very appealing property of the POD is its optimality. Among all possible decompositions of a random field, the POD is the most efficient in the sense that for a given number of modes, the projection on the subspace used for modelling the random field will on average contain the most energy possible. Although POD has been regularly applied to non-linear problems, it is essential to underline that it is a linear technique and that it is optimal only with respect to other linear representations. The applications of this procedure are extensive in modelling of turbulence [1, 2] and image processing [3], and POD is now emerging as a useful tool in the field of structural dynamics. For instance, it has been applied to estimate the dimensionality of a system [4], to build reduced order models [5, 6], and to the identification and updating of non-linear systems [7–9].

The purpose of this paper is to determine whether a physical interpretation can be attributed to the modes obtained from the decomposition, i.e., the proper orthogonal modes (POMs). Particularly, it is inquired when the POMs are related to the vibration eigenmodes. This work is closely related to the paper of Feeny and Kappagantu [10]. However, in the present paper, the emphasis is shifted towards the singular value decomposition of the displacement matrix rather than the eigenvalue problem of the covariance matrix. Furthermore, the case of linear systems under harmonic and white noise excitations is discussed in greater detail.

The paper is organized as follows. In Section 2, the POD is briefly introduced. Section 3 gives a brief review of the singular value decomposition and its properties that are relevant in the context of this paper. Sections 4, 5 and 6 study the physical interpretation of the POMs of discrete linear systems, respectively, for the free response in the undamped and damped cases, and for the harmonic response. Section 7 offers a geometric approach to the

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comparison between vibration eigenmodes and POMs. It also investigates the relationship between non-linear normal modes (NNMs) and POMs. Finally, the discussion of the stationary random response of a linear system to a white noise excitation is included in Appendix A.

2. PROPER ORTHOGONAL DECOMPOSITION

Proper orthogonal decomposition, also known as Karhunen–Loeve transform, was introduced by Kosambi [11]. It is also worth pointing out that POD is closely related to principal component analysis (PCA) introduced by Hotelling [12]. For a detailed historical review of POD or PCA, the reader is referred to references [10, 13].

Let $v(x, t)$ be a zero mean random field on a domain Ω. In practice, the field is sampled at a finite number of points in time. Then, at time t_i, the system displays a snapshot $v_i(x)$ which is a continuous function of x in Ω. The aim of the POD is to find the most persistent structure $\phi(x)$ among the ensemble of n snapshots. This is equivalent to minimizing the objective function λ:

$$\text{Minimize} \quad \lambda = \sum_{i=1}^{n} (\phi(x) - v_i(x))^2 \quad \forall x \in \Omega. \tag{1}$$

Equation (1) can also be written in terms of a maximization problem [6]:

$$\text{Maximize} \quad \left\{ \lambda = \frac{(1/N)\sum_{n=1}^{N} (\int_\Omega \phi(x) v_n(x)\,\mathrm{d}\Omega)^2}{\int_\Omega \phi(x)\phi(x)\,\mathrm{d}\Omega} \right\} \quad \forall x \in \Omega. \tag{2}$$

Finally, the optimization problem can be reduced to the following integral eigenvalue problem [6]:

$$\int_\Omega K(x, x')\phi(x')\,\mathrm{d}x' = \lambda\phi(x'), \tag{3}$$

where K is the two-point correlation function

$$K(x, x') = \frac{1}{n}\sum_{i=1}^{n} v_i(x) v_i(x'). \tag{4}$$

Equation (3) has a finite number of orthogonal solutions $\phi^i(x)$, called the proper orthogonal modes (POMs) with corresponding real and positive eigenvalues λ^i. In practice, the snapshots are available at discrete measurement points x_k where $k = 1, \ldots, m$ and the integral eigenvalue problem (3) reduces to find the eigensolution of an $(m \times m)$ space correlation tensor

$$\mathbf{G} = \begin{bmatrix} K(x_1, x_1) & \cdots & K(x_1, x_m) \\ \cdots & \cdots & \cdots \\ K(x_m, x_1) & \cdots & K(x_m, x_m) \end{bmatrix}. \tag{5}$$

To summarize, if the responses (e.g., the displacements) $q_k(t)$ of a discrete dynamical system with m degrees of freedom (d.o.f.) are sampled n times and if the $(m \times n)$ matrix

$$\mathbf{Q} = \begin{bmatrix} q_1(t_1) & \cdots & q_1(t_n) \\ \cdots & \cdots & \cdots \\ q_m(t_1) & \cdots & q_m(t_n) \end{bmatrix} = [\mathbf{q}(t_1) \cdots \mathbf{q}(t_m)] \tag{6}$$

is formed, then the POMs are merely the eigenvectors of $\mathbf{G} = (1/n)\mathbf{Q}\mathbf{Q}^{\mathrm{T}}$ and the corresponding eigenvalues are the proper orthogonal values (POVs). A POV measures the relative energy of the system dynamics contained in the associated POM.

3. SINGULAR VALUE DECOMPOSITION

The objective of this section is to review the singular value decomposition (SVD) and its features that are relevant in the context of POD. Particularly, it is pointed out that the POMs are optimal with respect to energy content. For a detailed description of SVD and its several possible applications in structural dynamics, the reader is referred to references [14, 15]. Since the matrices considered throughout the paper are built from system responses, e.g., displacements, the discussion is restricted to real matrices only.

For any real $(m \times n)$ matrix $\mathbf{A}$, there exists a real factorization

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}, \tag{7}$$

where $\mathbf{U}$ is an $(m \times m)$ orthonormal matrix. Its columns form the left singular vectors. $\boldsymbol{\Sigma}$ is an $(m \times n)$ pseudo-diagonal and semi-positive-definite matrix with diagonal entries containing the singular values σ_i. $\mathbf{V}$ is an $(n \times n)$ orthonormal matrix. Its columns form the right singular vectors.

3.1. GEOMETRIC INTERPRETATION

The SVD of a matrix, seen as a collection of column vectors, provides important insight into the oriented energy distribution of this set of vectors. It is worth recalling that

1. the energy of a vector sequence $\mathbf{a}_k$ building an $(m \times n)$ matrix $\mathbf{A}$ is defined via the Frobenius norm

$$E(\mathbf{A}) = \|\mathbf{A}\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 = \sum_{k=1}^{p} \sigma_k^2 \quad \text{where } p = \min(m, n), \tag{8}$$

   so that the energy of a vector sequence is equal to the energy in its singular spectrum;
2. the oriented energy of a vector sequence in some direction p with unit vector $\mathbf{e}_p$ of the m-dimensional column space is the sum of squared projections of the vectors on to direction p

$$E_p(\mathbf{A}) = \sum_{k=1}^{n} (\mathbf{e}_p^{\mathrm{T}}\mathbf{a}_k)^2. \tag{9}$$

One essential property of SVD is that extrema in this oriented energy distribution occur at each left singular direction [15]. The oriented energy measured in the direction of the ith left singular vector is equal to the ith singular value squared. Since the POMs are directly related to the left singular vectors, it can be stated that they are optimal with respect to energy content in a least-square sense, i.e., they capture more energy per mode than any other set of basis functions.

3.2. RELATION WITH THE EIGENVALUE PROBLEM

The SVD of a matrix can be calculated by means of solving two eigenvalue problems, or even one if only the left or the right singular vectors are required. Indeed,

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{U}\boldsymbol{\Sigma}^2\mathbf{U}^{\mathrm{T}} \qquad \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{V}\boldsymbol{\Sigma}^2\mathbf{V}^{\mathrm{T}}. \tag{10}$$

Consequently, the singular values of $\mathbf{A}$ are found to be the square roots of the eigenvalues of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ or $\mathbf{A}^{\mathrm{T}}\mathbf{A}$. The left and right singular vectors of $\mathbf{A}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ and $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ respectively. Applying this reasoning to POD, it is now clear that the POMs, defined as the eigenvectors of the covariance matrix $\mathbf{G} = (1/n)\mathbf{A}\mathbf{A}^{\mathrm{T}}$, are the left singular vectors of $\mathbf{A}$. The POVs, defined as the eigenvalues of the covariance matrix, are the square of the singular values divided by the number of samples n. In conclusion, POD can be carried out directly by means of an SVD of matrix $\mathbf{A}$.

An interesting interpretation of the eigenvalue problem is that if a matrix is real, symmetric and positive definite, then the eigenvectors of the matrix are the principal axes of the associated quadratic form which is an n-dimensional ellipsoid centered at the origin of the Euclidean space [16]. Since $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is real, symmetric and positive definite, the POMs as eigenvectors of the covariance matrix are the principal axes of the family of ellipsoids defined by $\mathbf{y}^{\mathrm{T}}\mathbf{G}\mathbf{y} = c$ where $\mathbf{y}$ is a real non-zero vector and c is a positive constant.

It is worth pointing out that Feeny and Kappagantu showed that if each data has unit mass, then the POMs are the principal axes of inertia [10].

4. UNDAMPED AND UNFORCED LINEAR SYSTEMS

The aim of this section is to find the existing relationships between the POMs and the eigenmodes of an undamped and unforced linear system with m. d.o.f. The equation of motion may be written as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0, \tag{11}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, respectively, and $\mathbf{q}$ is the vector of displacement co-ordinates.

The system response due to initial conditions may be expressed as

$$\mathbf{q}(t) = \sum_{i=1}^{m} (A_i \cos\omega_i t + B_i \sin\omega_i t)\mathbf{x}_{(i)} = \sum_{i=1}^{m} e_i(t)\mathbf{x}_{(i)}, \tag{12}$$

where ω_i, $\mathbf{x}_{(i)}$ are the natural frequencies (in rad/s) and eigenmodes of the system; A_i and B_i are constants depending on the initial conditions; and $e_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t$ represents the time modulation of mode $\mathbf{x}_{(i)}$.

The time discretization of the system response leads to n sampled values of the time functions which form an $(m \times n)$ matrix whose columns are the members of the data ensemble

$$\begin{aligned}\mathbf{Q} &= [\mathbf{q}(t_1) \cdots \mathbf{q}(t_n)] \\ &= \left[\sum_{i=1}^{m} e_i(t_1)\mathbf{x}_{(i)} \cdots \sum_{i=1}^{m} e_i(t_n)\mathbf{x}_{(i)} \right],\end{aligned} \tag{13}$$

which can also be written as

$$\begin{aligned}\mathbf{Q} &= [\mathbf{x}_{(1)} \cdots \mathbf{x}_{(m)}] \begin{bmatrix} e_1(t_1) & \cdots & e_1(t_n) \\ \cdots & & \cdots \\ e_m(t_1) & \cdots & e_m(t_n) \end{bmatrix} \\ &= [\mathbf{x}_{(1)} \cdots \mathbf{x}_{(m)}] \begin{bmatrix} \mathbf{e}_1^{\mathrm{T}} \\ \cdots \\ \mathbf{e}_m^{\mathrm{T}} \end{bmatrix} \\ &= [\mathbf{x}_{(1)} \cdots \mathbf{x}_{(m)}]\, [\mathbf{e}_1 \cdots \mathbf{e}_m]^{\mathrm{T}} \\ &= \mathbf{X}\mathbf{E}^{\mathrm{T}} \\ &= \mathbf{X}[\mathbf{I}\ \mathbf{Z}]\, [\mathbf{E}\ \mathbf{R}]^{\mathrm{T}},\end{aligned} \tag{14}$$

where $\mathbf{X}$ is the $(m \times m)$ modal matrix whose columns are the eigenmodes of the system; $\mathbf{E}$ is an $(n \times m)$ matrix whose columns are the functions $e_i(t)$ at times $t_1, \ldots, t_n$; $\mathbf{I}$ is an $(m \times m)$ identity matrix; $\mathbf{Z}$ is an $(m \times (n-m))$ matrix full of zeros; $\mathbf{R}$ is an $(n \times (n-m))$ matrix; and $\mathbf{e}_i = [e_i(t_1) \cdots e_i(t_n)]^{\mathrm{T}}$.

Attention should be paid to the fact that $\mathbf{R}$ does not influence $\mathbf{Q}$ since it is multiplied by a matrix full of zeros. Equation (14) can be expressed in a more familiar form as

$$\mathbf{Q} = \underbrace{[\mathbf{X}]}_{\mathbf{U}}\ \underbrace{[\mathbf{I}\ \mathbf{Z}]}_{\mathbf{\Sigma}}\ \underbrace{[\mathbf{E}\ \mathbf{R}]^{\mathrm{T}}}_{\mathbf{V}^{\mathrm{T}}} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}. \tag{15}$$

Accordingly, the above decomposition of $\mathbf{Q}$ may be thought of as the SVD of this matrix. However, this decomposition requires matrices $\mathbf{U}$ and $\mathbf{V}$ to be orthonormal as mentioned in Section 3. The aim now is to find the conditions when the columns of $\mathbf{U}(\equiv \mathbf{X})$ and $\mathbf{V}(\equiv [\mathbf{E}\ \mathbf{R}])$ are orthogonal.

1. The columns of $\mathbf{U}$ are formed by the eigenmodes of the structure. The eigenmodes are orthogonal to each other in the metrics of the mass and stiffness matrices. If the mass matrix is proportional to the identity matrix, it turns out that $\mathbf{x}_{(i)}^{\mathrm{T}}\mathbf{x}_{(j)} = \delta_{ij}$. Consequently, $\mathbf{X}$ is orthogonal if the mass matrix is proportional to the identity matrix.

2. It remains to determine when the columns of $\mathbf{V}$ are orthogonal. For this purpose, equation (14) may be rewritten as follows:

$$\mathbf{Q} = \mathbf{X}[\mathbf{I}\ \mathbf{Z}]\ [\mathbf{E}\ \mathbf{R}]^{\mathrm{T}} \tag{16}$$

$$= \mathbf{X}[\mathrm{diag}(\|\mathbf{e}_i\|)\ \mathbf{Z}]\ [\mathbf{E}\,\mathrm{diag}(\|\mathbf{e}_i\|^{-1})\ \mathbf{R}]^{\mathrm{T}}$$

$$= [\mathbf{x}_{(1)} \cdots \mathbf{x}_{(m)}] \begin{bmatrix} \|\mathbf{e}_1\| & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \|\mathbf{e}_2\| & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \|\mathbf{e}_m\| & 0 & \cdots & 0 \end{bmatrix} \left[\frac{\mathbf{e}_1}{\|\mathbf{e}_1\|}\ \frac{\mathbf{e}_2}{\|\mathbf{e}_2\|} \cdots \frac{\mathbf{e}_m}{\|\mathbf{e}_m\|}\ \mathbf{R} \right]^{\mathrm{T}}.$$

If the natural frequencies ω_i are distinct, it can be easily argued that the columns of $\mathbf{E}\ \mathrm{diag}(\|\mathbf{e}_i\|^{-1})$ are orthogonal if we consider an infinite set of sampled values, i.e.,

$$\frac{\mathbf{e}_i}{\|\mathbf{e}_i\|}\ \frac{\mathbf{e}_j}{\|\mathbf{e}_j\|} \to 0 \quad \text{if } n \to \infty,\ i \neq j. \tag{17}$$

Since $\mathbf{R}$ does not have an influence on $\mathbf{Q}$, its columns can be computed in order that they are orthogonal to those of $\mathbf{E}\,\mathrm{diag}(\|\mathbf{e}_i\|^{-1})$. As can also be seen from equation (16), POD is a bi-orthogonal decomposition that uncouples the spatial and temporal information contained in the data.

To summarize, if the mass matrix is proportional to the identity matrix and if the number of samples is infinite, the singular value decomposition of $\mathbf{Q}$ is such that

(1) the columns of $\mathbf{U}$ are the eigenmodes;
(2) the first n columns of $\mathbf{V}$ are the normalized time modulations of the modes.

As stated in section 3.2, the POD basis vectors are just the columns of the matrix $\mathbf{U}$ in the singular value decomposition of the displacement matrix. Therefore, it can be concluded that the POMs converge to the eigenmodes of an undamped and unforced linear system whose mass matrix is proportional to identity if a *sufficient* number of samples is considered. Feeny and Kappagantu [10] previously obtained the same conclusion by a different way. They based their demonstration on the fact that the POMs are the eigenvectors of the covariance matrix.

In the case of a mass matrix not proportional to identity, the POMs no longer converge to the eigenmodes since the former are orthogonal to each other while the latter are orthogonal with respect to the mass matrix. However, knowing the mass matrix, it is still possible to retrieve the eigenmodes from the POMs. Equation (11) has to be rewritten through the co-ordinate transformation $\mathbf{q} = \mathbf{M}^{-1/2}\mathbf{p}$ as

$$\ddot{\mathbf{p}} + \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{p} = 0. \tag{18}$$

In equation (18), the system matrices are still symmetric while the effective mass matrix is equal to the identity. Thus, the left singular vectors of $\mathbf{P} = [\mathbf{p}(t_1) \cdots \mathbf{p}(t_n)]$, i.e., the POMs, converge to the eigenmodes $\mathbf{y}_{(i)}$ of this system. It is a simple matter to demonstrate that the eigenmodes $\mathbf{x}_{(i)}$ of system (11) are related to those of system (18) by the following relationship:

$$\mathbf{x}_{(i)} = \mathbf{M}^{-1/2}\mathbf{y}_{(i)}. \tag{19}$$

This section has investigated the discrete case. A detailed study of distributed systems can be found in reference [17]. This paper underlines that the conclusions are still valid if the distributed system is uniformly discretized.

5. DAMPED AND UNFORCED LINEAR SYSTEMS

Consider now a damped but still unforced linear system with m. d.o.f. for which the equation of motion is given as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0. \tag{20}$$

If the structure is lightly damped or with the assumption of modal damping, the system response can be readily written as

$$\mathbf{q}(t) = \sum_{i=1}^{m} A_i \exp^{-\varepsilon_i \omega_i t} \cos(\sqrt{1-\varepsilon_i^2}\,\omega_i t + \alpha_i)\mathbf{x}_{(i)} = \sum_{i=1}^{m} e_i(t)\mathbf{x}_{(i)}. \tag{21}$$

Using the same procedure as in the previous section yields

$$\begin{aligned}
\mathbf{Q} &= [\mathbf{q}(t_1)\cdots\mathbf{q}(t_n)] \\
&= \left[\sum_{i=1}^{m} e_i(t_1)\mathbf{x}_{(i)} \cdots \sum_{i=1}^{m} e_i(t_n)\mathbf{x}_{(i)}\right] \\
&= [\mathbf{x}_{(1)}\cdots\mathbf{x}_{(m)}]\begin{bmatrix}\mathbf{e}_1^{\mathrm{T}} \\ \cdots \\ \mathbf{e}_m^{\mathrm{T}}\end{bmatrix} \\
&= \mathbf{X}\mathbf{E}^{\mathrm{T}} \\
&= \mathbf{X}[\mathbf{I}\ \mathbf{Z}]\ [\mathbf{E}\ \mathbf{R}]^{\mathrm{T}} \\
&= \mathbf{X}[\mathrm{diag}(\|\mathbf{e}_i\|)\ \mathbf{Z}]\ [\mathbf{E}\,\mathrm{diag}(\|\mathbf{e}_i\|^{-1})\ \mathbf{R}]^{\mathrm{T}} \\
&= [\mathbf{x}_{(1)}\cdots\mathbf{x}_{(m)}]\begin{bmatrix}\|\mathbf{e}_1\| & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \|\mathbf{e}_2\| & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \|\mathbf{e}_m\| & 0 & \cdots & 0\end{bmatrix}\left[\frac{\mathbf{e}_1}{\|\mathbf{e}_1\|}\ \frac{\mathbf{e}_2}{\|\mathbf{e}_2\|}\cdots\frac{\mathbf{e}_m}{\|\mathbf{e}_m\|}\ \mathbf{R}\right]^{\mathrm{T}}. \\
&= \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}},
\end{aligned} \tag{22}$$

where $\mathbf{e}_i = [A_i \exp^{-\varepsilon_i\omega_i t}\cos(\sqrt{1-\varepsilon_i^2}\,\omega_i t_1 + \alpha_i)\cdots A_i \exp^{-\varepsilon_i\omega_i t}\cos(\sqrt{1-\varepsilon_i^2}\,\omega_i t_n + \alpha_i)]^{\mathrm{T}}$.

Again, the columns of $\mathbf{U}(\equiv\mathbf{X})$ are orthogonal if the mass matrix is proportional to the identity matrix. The main difference with the undamped case is that the time modulations $e_i(t)\to 0$ if $t\to\infty$ since the system returns to the equilibrium position in a finite time. Consequently, it can no longer be affirmed that $\|\mathbf{e}_i\|\to\infty$ if $n\to\infty$ and that the columns of $\mathbf{E}\,\mathrm{diag}(\|\mathbf{e}_i\|^{-1})$ are orthogonal to each other. This causes a set of POMs different from the eigenmodes to be obtained. However, if the damping is low and if a sufficient number of points are considered, $\mathbf{E}\,\mathrm{diag}(\|\mathbf{e}_i\|^{-1})$ is almost orthogonal. In conclusion, the POMs of a lightly damped and unforced linear system are a very good approximation of the eigenmodes of this system. This is in accordance with the result obtained in reference [10] using the eigensolution perspective.

6. HARMONIC AND FORCED HARMONIC RESPONSES OF A LINEAR SYSTEM

This section is divided into two parts. Firstly, the harmonic response of a linear system is considered. By harmonic response, we mean the combination of the free and forced responses. Secondly, attention is focused only on the forced response of the linear system.

6.1. HARMONIC RESPONSE

The equation of motion of a linear system with m. d.o.f. excited by an harmonic force with a constant amplitude is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}\sin\omega_e t. \tag{23}$$

Equation (23) may be transformed by considering a new variable $s = \sin\omega_e t$ that accounts for the harmonic force

$$\begin{matrix} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}s \\ \ddot{s} + \omega_e^2 s = 0 \end{matrix} \quad \text{with } s(0) = 0, \quad \dot{s}(0) = \omega_e, \tag{24}$$

which yields

$$\overbrace{\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}}^{\mathbf{M}^*} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{s} \end{bmatrix} + \overbrace{\begin{bmatrix} \mathbf{K} & -\mathbf{f} \\ \mathbf{0} & \omega_e^2 \end{bmatrix}}^{\mathbf{K}^*} \begin{bmatrix} \mathbf{q} \\ s \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}. \tag{25}$$

For the sake of clarity, note $(\omega_i^2, \mathbf{x}_{(i)})$ the eigensolutions of the initial system (23) and $(\omega_i^{*2}, \mathbf{x}^*_{(i)})$ the eigensolutions of the transformed system (24). This latter system may be viewed as an unforced system with $m+1$ d.o.f. (25). If the mass matrix is proportional to identity and if the number of samples is large enough, section 4 allows us to conclude that the POMs of the transformed system response converge to the eigenmodes of that system.

Let us now compute the eigenmodes of the transformed system. These are the solution of

$$(\mathbf{K}^* - \omega_i^{*2}\mathbf{M}^*)\mathbf{x}^*_{(i)} = 0 \tag{26}$$

if ω_i^{*2} is a root of the algebraic equation

$$\det(\mathbf{K}^* - \omega^2\mathbf{M}^*) = \det\left(\begin{bmatrix} \mathbf{K} & -\mathbf{f} \\ \mathbf{0} & \omega_e^2 \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) = 0. \tag{27}$$

This equation becomes

$$\det\left(\begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & -\mathbf{f} \\ \mathbf{0} & \omega_e^2-\omega^2 \end{bmatrix}\right) = (\omega_e^2-\omega^2)\det(\mathbf{K}-\omega^2\mathbf{M}) = 0. \tag{28}$$

As can be seen from equation (28), the transformed system has $m+1$ eigenvalues. m eigenvalues are equal to those of the initial system (23)

$$\omega_i^{*2} = \omega_i^2 \quad \text{with } i = 1,\ldots,m \tag{29}$$

and the additional eigenvalue is equal to the square of the excitation frequency (in rad/s)

$$\omega_{m+1}^{*2} = \omega_e^2. \tag{30}$$

The eigenmodes corresponding to these eigenvalues may now be calculated. As illustrated in equation (26), the eigenmodes are the solution of

$$(\mathbf{K}^*-\omega_i^{*2}\mathbf{M}^*)\mathbf{x}_{(i)}^* = \begin{bmatrix} \mathbf{K}-\omega_i^{*2}\mathbf{M} & -\mathbf{f} \\ \mathbf{0} & \omega_e^2-\omega_i^{*2} \end{bmatrix}\mathbf{x}_{(i)}^* = 0. \tag{31}$$

For $\omega_i^{*2} = \omega_i^2$, an obvious solution of system (31) is

$$\mathbf{x}_{(i)}^* = \begin{bmatrix} \mathbf{x}_{(i)} \\ 0 \end{bmatrix}. \tag{32}$$

Accordingly, the eigenmodes of the transformed system corresponding to ω_i^2 have the first m components equal to the eigenmodes of the initial system. The last component is equal to 0.

It remains to evaluate the eigenmode corresponding to ω_e^2, i.e., $\mathbf{x}_{(m+1)}^*$. With this aim, finding the eigensolutions of the transformed system is also equivalent to finding the eigensolutions of matrix

$$\mathbf{M}^{*-1}\mathbf{K}^* = \begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{K} & -\mathbf{f} \\ \mathbf{0} & \omega_e^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{f} \\ \mathbf{0} & \omega_e^2 \end{bmatrix} \tag{33}$$

and

$$\begin{bmatrix} \mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{f} \\ \mathbf{0} & \omega_e^2 \end{bmatrix}[\mathbf{x}_{(1)}^* \cdots \mathbf{x}_{(m)}^* \ \mathbf{x}_{(m+1)}^*]$$

$$= [\mathbf{x}_{(1)}^* \cdots \mathbf{x}_{(m)}^* \ \mathbf{x}_{(m+1)}^*]\begin{bmatrix} \operatorname{diag}(\omega_i^{*2},\ldots,\omega_m^{*2}) & \mathbf{0} \\ \mathbf{0} & \omega_{m+1}^{*2} \end{bmatrix} \tag{34}$$

$$\left[\begin{array}{c|c} \mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{f} \\ \hline \mathbf{0} & \omega_e^2 \end{array}\right]\left[\begin{array}{c|c} \mathbf{X} & \mathbf{x}_{(m+1),1:m}^* \\ \hline \mathbf{0} & x_{(m+1),m+1}^* \end{array}\right] = \left[\begin{array}{c|c} \mathbf{X} & \mathbf{x}_{(m+1),1:m}^* \\ \hline \mathbf{0} & x_{(m+1),m+1}^* \end{array}\right]\left[\begin{array}{c|c} \Omega & \mathbf{0} \\ \hline \mathbf{0} & \omega_e^2 \end{array}\right], \tag{35}$$

where $\mathbf{X} = [\mathbf{x}_{(1)} \cdots \mathbf{x}_{(m)}]$ and $\Omega = \mathrm{diag}(\omega_1^2, \ldots, \omega_m^2)$ are the eigensolutions of the initial system. It follows from equation (35) that

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{X} = \mathbf{X}\mathbf{\Omega}, \tag{36}$$

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{x}^*_{(m+1),1:m} - \mathbf{M}^{-1}\mathbf{f}x^*_{(m+1),m+1} = \mathbf{x}^*_{(m+1),1:m}\omega_e^2, \tag{37}$$

$$\mathbf{0} = \mathbf{0}, \tag{38}$$

$$\omega_e^2 x^*_{(m+1),m+1} = x^*_{(m+1),m+1}\omega_e^2. \tag{39}$$

Equation (37) allows us to calculate the first m components of the last eigenmode $\mathbf{x}^*_{(m+1)}$:

$$\mathbf{x}^*_{(m+1),1:m} = [\mathbf{M}^{-1}\mathbf{K} - \omega_e^2\mathbf{I}]^{-1}\mathbf{M}^{-1}\mathbf{f}x^*_{(m+1),m+1} = [\mathbf{K} - \omega_e^2\mathbf{M}]^{-1}\mathbf{f}x^*_{(m+1),m+1}. \tag{40}$$

$[\mathbf{K} - \omega_e^2\mathbf{M}]^{-1}$ is the dynamic influence coefficient matrix and its spectral expansion is [18]

$$[\mathbf{K} - \omega_e^2\mathbf{M}]^{-1} = \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i}. \tag{41}$$

Let us now introduce the spectral expansion (41) in equation (40)

$$\mathbf{x}^*_{(m+1),1:m} = \left\{ \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i} \right\} \mathbf{f}x^*_{(m+1),m+1}. \tag{42}$$

Since an eigenmode is defined as a scale factor and since $x^*_{(m+1),m+1}$ is a scalar, the final expression for the eigenmode corresponding to ω_e^2 is

$$\mathbf{x}^*_{(m+1),1:m} = \left\{ \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i} \right\} \mathbf{f}. \tag{43}$$

To summarize, consider a matrix which contains the response of the transformed system (24), i.e., its first m rows contain the response of the initial system (23) and its $(m + 1)$th row is the applied force

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{q}(t_1) \cdots \mathbf{q}(t_n) \\ s(t_1) \cdots s(t_n) \end{bmatrix}. \tag{44}$$

This matrix has $m + 1$ POMs that have $m + 1$ components. The dominant POM is related to the forced harmonic response of the system and its first m components are given by equation (43). Furthermore, if the mass matrix is proportional to identity, the first m components of the remaining POMs are merely the eigenmodes of the linear system. This perspective should be useful in the context of modal analysis.

6.2. FORCED HARMONIC RESPONSE

The forced response is defined as the part of the response synchronous to the excitation

$$\mathbf{q}(t) = \mathbf{q}_f \sin \omega_e t. \tag{45}$$

The forced response amplitude is described by the following expression [18]:

$$\mathbf{q}_f = \left\{ \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i} \right\} \mathbf{f} \tag{46}$$

and

$$\mathbf{q}(t) = \left\{ \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i} \right\} \mathbf{f} \sin \omega_e t. \tag{47}$$

The displacement matrix $\mathbf{Q}$ becomes

$$\begin{aligned} \mathbf{Q} &= [\mathbf{q}(t_1) \cdots \mathbf{q}(t_n)] \\ &= \left[\left\{ \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i} \right\} \mathbf{f} \sin \omega_e t_1 \cdots \left\{ \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i} \right\} \mathbf{f} \sin \omega_e t_n \right]. \end{aligned} \tag{48}$$

Equation (48) may be expressed in the form

$$\begin{aligned} \mathbf{Q} &= \left[\left\{ \sum_{i=1}^{m} \frac{\mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}}{(\omega_i^2 - \omega_e^2)\mu_i} \right\} \mathbf{f} \right] \begin{bmatrix} \sin \omega_e t_1 \\ \cdots \\ \sin \omega_e t_n \end{bmatrix}^{\mathrm{T}} \\ &= \mathbf{q}_f \mathbf{e}^{\mathrm{T}} \\ &= [\mathbf{q}_f \;\; \mathbf{S} \;\;] \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots & & \vdots \\ \vdots & & \ddots & & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} [\mathbf{e} \;\; \mathbf{R}]^{\mathrm{T}} \\ &= \left[\frac{\mathbf{q}_f}{\|\mathbf{q}_f\|} \;\; \mathbf{S} \right] \begin{bmatrix} \|\mathbf{q}_f\| \, \|\mathbf{e}\| & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots & & \vdots \\ \vdots & & \ddots & & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \left[\frac{\mathbf{e}}{\|\mathbf{e}\|} \;\; \mathbf{R} \right]^{\mathrm{T}} \\ &= \underbrace{\left[\frac{\mathbf{q}_f}{\|\mathbf{q}_f\|} \;\; \mathbf{S} \right]}_{\mathbf{U}} \underbrace{[\mathbf{I}_1]}_{\boldsymbol{\Sigma}} \underbrace{\left[\frac{\mathbf{e}}{\|\mathbf{e}\|} \;\; \mathbf{R} \right]^{\mathrm{T}}}_{\mathbf{V}^{\mathrm{T}}}, \end{aligned} \tag{49}$$

where $\mathbf{S}$ is an $(m \times (m-1))$ matrix, $\mathbf{I}_1$ is an $(m \times n)$ matrix containing only one non-zero element $\|\mathbf{q}_f\| \, \|\mathbf{e}\|$, and $\mathbf{R}$ is an $(n \times (n-1))$ matrix.

Matrices $\mathbf{S}$ and $\mathbf{R}$ do not influence equation (49) since they are both multiplied by zero elements. If $\mathbf{S}$ and $\mathbf{R}$ are chosen in order that $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, then equation (49) is the singular value decomposition of the matrix $\mathbf{Q}$.

In conclusion, the following points may be noted.

1. Since there is only one non-zero singular value, the forced harmonic response of a linear system is captured by a single POM whatever the number of d.o.f. is. Nevertheless, all the eigenmodes are necessary to reconstruct the response. This property is independent of the mass distribution and underlines the optimality of the POMs described in section 3.1.
2. An analytical expression of the POM is obtained:

$$POM = \frac{\{\sum_{i=1}^{m} \mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}/(\omega_i^2 - \omega_e^2)\mu_i\}\mathbf{f}}{\|\{\sum_{i=1}^{m} \mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}/(\omega_i^2 - \omega_e^2)\mu_i\}\mathbf{f}\|}. \tag{50}$$

   Knowing the structural matrices and the spatial discretization of the excitation, the POM may be calculated without first simulating the system response as required in the definition of the POMs.
3. The expression of the POM (50) is equal, to the norm, to the last eigenmode of the transformed system for the harmonic response (43). This last eigenmode is thus related to the forced harmonic response.
4. The convergence of the dominant POM to an eigenmode is no longer guaranteed. The POM appears now as a combination of all the eigenmodes. However, if the excitation frequency ω_e tends to a resonant frequency of the system, ω_j for instance, then the denominator $\omega_j^2 - \omega_e^2$ of the jth term of combination (50) tends to zero. It is thus observed that this term has a much larger amplitude than the others:

$$POM = \frac{\{\sum_{i=1}^{m} \mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}/(\omega_i^2 - \omega_e^2)\mu_i\}\mathbf{f}}{\|\{\sum_{i=1}^{m} \mathbf{x}_{(i)}\mathbf{x}_{(i)}^{\mathrm{T}}/(\omega_i^2 - \omega_e^2)\mu_i\}\mathbf{f}\|} \simeq \frac{\mathbf{x}_{(j)}\mathbf{x}_{(j)}^{\mathrm{T}}/(\omega_j^2 - \omega_e^2)\mu_j\mathbf{f}}{\|\mathbf{x}_{(j)}\mathbf{x}_{(j)}^{\mathrm{T}}/(\omega_j^2 - \omega_e^2)\mu_j\mathbf{f}\|}$$
$$= \alpha x_{(j)} \quad \text{if } \omega_e \to \omega_j. \tag{51}$$

   Since $\mathbf{x}_{(j)}^{\mathrm{T}}\mathbf{f}$ represents a scalar product, the POM has the same direction as the eigenmode $\mathbf{x}_{(j)}$ which means that the POM is equal to the resonating mode shape. This is consistent with that obtained in reference [10] using the eigensolution perspective. It is worth pointing out that the non-resonating mode shapes should not be revealed by POD.

7. LINEAR NORMAL MODES, NON-LINEAR NORMAL MODES AND PROPER ORTHOGONAL MODES: A GEOMETRIC APPROACH

For the sake of clarity, the eigenmodes of a linear system are called here linear normal modes (LNMs). The determination of LNMs is reduced to the equivalent problem of computing the eigensolutions of linear transformations. Obviously, such an approach as well as the superposition principle is inadmissible for non-linear systems. The concept of synchronous non-linear normal mode (NNM) for discrete conservative oscillators was introduced for non-linear systems by Rosenberg [19]: “A nonlinear system vibrates in normal modes when all masses execute periodic motions of the same period, when all of them pass through the equilibrium position at the same instant, and when, at any time t, the position of all the masses is uniquely defined by the position of any one of them.”

The objective of this section is to examine the geometric interpretation of LNMs, NNMs and POMs.

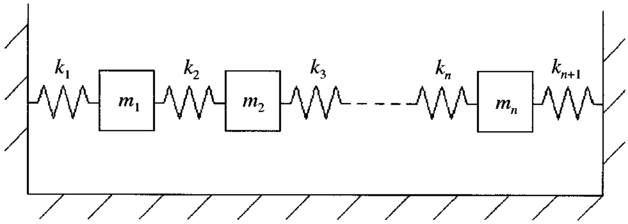

Figure 1. Linear system consisting of masses and springs.

7.1. LINEAR SYSTEMS

Consider a linear system consisting of masses and springs (Figure 1). If the displacement of the ith mass from its equilibrium position is denoted by q_i, then the equations of motion of the system are

$$m_i\ddot{q}_i = k_i(q_{i-1} - q_i) - k_{i+1}(q_i - q_{i+1}) \quad \text{where } i = 1, 2, \ldots, n, \quad q_0 = q_{n+1} \equiv 0. \tag{52}$$

If the co-ordinates are normalized using the transformations $\xi_i = m_i^{1/2} q_i,\ i = 1, \ldots, n$, equation (52) becomes

$$\ddot{\xi}_i = \frac{k_i}{m_i^{1/2}}\left(\frac{\xi_{i-1}}{m_{i-1}^{1/2}} - \frac{\xi_i}{m_i^{1/2}}\right) - \frac{k_{i+1}}{m_i^{1/2}}\left(\frac{\xi_{i+1}}{m_{i+1}^{1/2}} - \frac{\xi_i}{m_i^{1/2}}\right) \quad \text{where } m_0 = m_{n+1} \equiv \infty. \tag{53}$$

The transformed equations of motion (53) may be regarded as those of a unit mass which moves in an n-dimensional space. The right-hand side of equation (53) derives from a potential function

$$\ddot{\xi}_i = \frac{\partial U}{\partial \xi_i}, \quad \text{with } U = -\sum_{i=1}^{n+1} \frac{k_i}{2}\left(\frac{\xi_{i-1}}{m_{i-1}^{1/2}} - \frac{\xi_i}{m_i^{1/2}}\right)^2. \tag{54}$$

If no external force is present and if the motion is due to an initial displacement, the system occupies at time $t = 0$ a position of maximum potential $U = -U_0$. This latter equation defines an ellipsoid which is symmetric with respect to the origin. This ellipsoid is called the bounding ellipsoid because all solutions must lie in this domain.

In its definition of a normal mode for a linear system, Rosenberg [19] stated that it is a straight line in the $(\xi_1, \ldots, \xi_n)$ space which passes through the origin of that space and which intersects the bounding ellipsoid orthogonally. It follows from the definition that the LNMs are the principal axes of the bounding ellipsoid in the $(\xi_1, \ldots, \xi_n)$ space. This result can also be obtained with the interpretation of the eigenvalue problem (section 3.2). Further discussion is given in Appendix B.

If the mass matrix is proportional to identity, the LNMs are also the principal axes of the bounding ellipsoid in the $(q_1, \ldots, q_n)$ space whose expression is

$$U_0 = \sum_{i=1}^{n+1} \frac{k_i}{2}(q_{i-1} - q_i)^2 = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}. \tag{55}$$

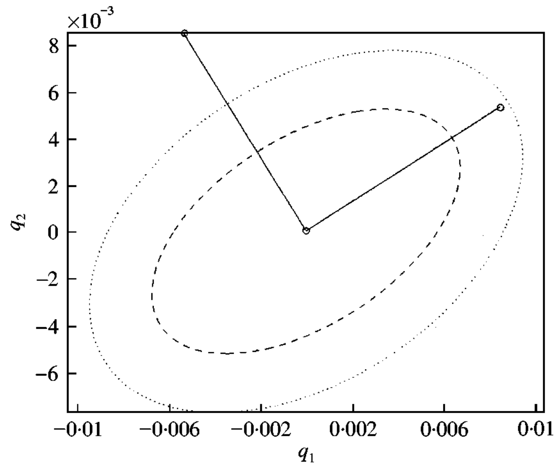

Figure 2. LNMs and POMs. Principal axes of similar and similarly placed ellipsoids: – –, K; ·····; G; ——, POM; ◇, LNM.

As far as the POMs are concerned, they are the principal axes of the ellipsoid $c = \mathbf{q}^{\mathrm{T}}\mathbf{G}\mathbf{q}$ where $\mathbf{G}$ is the covariance matrix (cf. Section 3.2). Since for an unforced system with a mass matrix proportional to identity, the POMs and the LNMs coincide, it can be concluded that $U_0 = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}$ and $c = \mathbf{q}^{\mathrm{T}}\mathbf{G}\mathbf{q}$ are similar and similarly placed ellipsoids. This is illustrated in Figure 2 (two d.o.f. system with an initial displacement).

7.2. NON-LINEAR SYSTEMS

If an LNM is a straight line in the co-ordinate space, an NNM is represented by a line, straight (similar NNM) or curved (non-similar NNM). But generally, NNMs are non-similar and the POMs, characterized by straight lines in the co-ordinate space, cannot be merged with NNMs. However, due to their optimality property if the motion is a single, synchronous NNM, the resonant POM minimizes the square of the distance with the NNM under the constraint that it passes through the origin of the co-ordinate system and as stated in reference [10], the POM can be considered as the best linear representation of the NNM. It is also worth pointing out that the LNMs are the tangent space to the NNMs [20].

8. CONCLUSION

This paper has presented a new way, based on the singular value decomposition, of interpreting the POMs in the field of structural dynamics. This work has underlined some features of POD which might be useful in the future. Since the POMs are related to the vibration eigenmodes in some cases, POD should be an alternative way of modal analysis for extracting the mode shapes of a structure. POMs could also be used to reconstruct a signal using a minimum number of modes.

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REFERENCES

1. P. HOLMES, J. L. LUMLEY and G. BERKOOZ 1996 *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge: New York.
2, W. CAZEMIER 1997 *Ph.D. Thesis, Rijksuniversiteit, Groningen*. Proper orthogonal decomposition and low dimensional models for turbulent flows.
3. G. UYTTERHOEVEN 1999 *Ph.D. Thesis, Katholieke Universiteit, Leuven*. Wavelets: software and applications.
4. J. P. CUSUMANO, M. T. SHARKADY and B. W. KIMBLE 1993 *Aerospace Structures: Nonlinear Dynamics and System Response, American Society of Mechanical Engineers* **AD-33**, 13–22. Spatial coherence measurements of a chaotic flexible-beam impact oscillator.
5. R. KAPPAGANTU and B. F. FEENY 1999 *Journal of Sound and Vibration* **224**, 863–877. An optimal modal reduction of a system with frictional excitation.
6. M. F. A. AZEEZ and A. F. VAKAKIS 1998 *Technical Report, University of Illinois at Urbana Champaign*. Proper orthogonal decomposition of a class of vibroimpact oscillations.
7. T. K. HASSELMAN, M. C. ANDERSON and W. G. GAN 1998 *Proceedings of the 16th International Modal Analysis Conference, Santa Barbara U.S.A.*, 644–651. Principal component analysis for nonlinear model correlation, updating and uncertainty evaluation.
8. V. LENAERTS, G. KERSCHEN and J. C. GOLINVAL 2000 *Proceedings of the 18th International Modal Analysis Conference, San Antonio, U.S.A.* Parameter identification of nonlinear mechanical systems using proper orthogonal decomposition.
9. V. LENAERTS, G. KERSCHEN and J. C. GOLINVAL 2001 *Mechanical Systems and Signal Processing* **15**, 31–43. Proper orthogonal decomposition for model updating of non-linear mechanical systems.
10. B. F. FEENY and R. KAPPAGANTU 1998 *Journal of Sound and Vibration* **211**, 607–616. On the physical interpretation of proper orthogonal modes in vibrations.
11. D. KOSAMBI 1943 *Journal of Indian Mathematical Society* **7**, 76–88. Statistics in function space.
12. H. HOTELLING 1933 *Journal of Educational Psychology* **24**, 417–441 and 498–520. Analysis of a complex of statistical variables into principal components.
13. B. RAVINDRA 1999 *Journal of Sound and Vibration* **219**, 189–192. Comments on "On the physical interpretation of proper orthogonal modes in vibrations".
14. D. OTTE 1994 *Ph.D. Thesis, Katholieke Universiteit, Leuven*. Development and evaluation of singular value analysis methodologies for studying multivariate noise and vibration problems.
15. J. STAAR 1982 *Ph.D. Thesis, Katholieke Universiteit, Leuven*. Concepts for reliable modelling of linear systems with application to on-line identification of multivariate state space descriptions.
16. L. MEIROVITCH 1980 *Computational Methods in Structural Dynamics*. Alphen a/d Rijn: Sijthoff and Noordhoff.
17. B. F. FEENY 1997 *Proceedings of ASME Design Engineering Technical Conferences, Sacramento, U.S.A.* Interpreting proper orthogonal modes in vibrations.
18. M. GERADIN and D. RIXEN 1994 *Mechanical Vibrations, Theory and Application to Structural Dynamics*. Paris: Masson.
19. R. M. ROSENBERG 1962 *Journal of Applied Mechanics* **29**, 7–14. The normal modes of nonlinear n-degree-of-freedom systems.
20. S. W. SHAW and C. PIERRE 1993 *Journal of Sound and Vibration* **164**, 85–124. Normal modes for non-linear vibratory systems.
21. A. E. BRYSON and Y. C. HO 1975 *Applied Optimal Control (Optimization, Estimation and Control)*. New York: Wiley.
22. D. F. MORRISON 1967 *Multivariate Statistical Methods*, McGraw-Hill Series in Probability and Statistics. New York: McGraw-Hill.

APPENDIX A: STATIONARY RANDOM RESPONSE OF A LINEAR SYSTEM TO A WHITE NOISE EXCITATION

This study concerns linear systems subjected to white noise sequences. With this aim, the equation of motion is recast in the state variable from

$$\dot{\mathbf{r}} = \mathbf{Ar} + \mathbf{Bw}, \qquad \mathbf{q} = \mathbf{Dr}, \tag{A1, A2}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{Z} & \mathbf{I} \\ \mathbf{M}^{-1}\mathbf{K} & \mathbf{M}^{-1}\mathbf{C} \end{bmatrix}$$

is the system matrix, $\mathbf{B}$ is the input matrix, $\mathbf{D}$ is the output matrix, and $\mathbf{w}(t)$ is a vector white noise process such that

$$E[\mathbf{w}(t)] = 0 \text{ and } E[\mathbf{w}(t)\mathbf{w}(\tau)^{\mathrm{T}}] = v\delta(t-\tau).$$

It is assumed that the system is stable and time invariant, and that all processes are Gaussian. In this context, it can be shown [21] that the covariance matrix of the steady state response $\mathbf{G}_r = E[\mathbf{r}(t)\mathbf{r}(t)^{\mathrm{T}}]$ satisfies the Lyapunov equation

$$\mathbf{AG}_r + \mathbf{G}_r\mathbf{A}^{\mathrm{T}} + \mathbf{B}v\mathbf{B}^{\mathrm{T}} = 0. \tag{A3}$$

It is worth pointing out that $\mathbf{G}_r$ also corresponds to the constant v in the definition of the controllability grammian $\mathbf{W}_c$ of the system.

If only the displacements are considered, then the covariance matrix of the system response becomes

$$\mathbf{G}_q = E[\mathbf{q}(t)\mathbf{q}(t)^{\mathrm{T}}] = \mathbf{DG}_r\mathbf{D}^{\mathrm{T}}. \tag{A4}$$

Equation (59) means that the POMs may be evaluated without first simulating the system. Indeed, if the structural matrices are assumed to be known, the Lyapunov equation (58) may be solved in order to compute the covariance matrix $\mathbf{G}_r$ and consequently $\mathbf{G}_q$. The POMs are then the eigenvectors of $\mathbf{G}_q$. The analytical relationship between the POMs and the eigenmodes is now obscured.

If all states (displacement and velocity) are measured, the POMs are merely the eigenvectors of the controllability grammian $\mathbf{W}_c$.

If the white noise excitation is Gaussian, then the POMs may be geometrically interpreted. In that case, the response of the linear system is also Gaussian and is characterized at each d.o.f. by a probability density function equal to

$$\theta(q) = \frac{1}{\sqrt{2\pi\sigma}} \exp^{[-\frac{1}{2}((q-\mu)/\sigma)^2]}, \tag{A5}$$

where $\mu = E[q]$ is the mean and $\sigma = E[(q-\mu)^2]$ is the standard deviation. The joint probability density function reads

$$\theta(q_1, \ldots, q_m) = \frac{1}{(2\pi)^{m/2}\sigma_1 \cdots \sigma_m} \exp^{[-\frac{1}{2}\sum_{i=1}^{m}((q_i-\mu_i)/\sigma_i)^2]}. \tag{A6}$$

It can be demonstrated [22] that the contours of $\theta(q_1, \ldots, q_m)$ consist of m-dimensional ellipsoids and that the POMs are the principal axes of these ellipsoids.

APPENDIX B: LNMs AND PRINCIPAL AXES OF THE BOUNDING ELLIPSOID

The LNMs are the eigenvectors of the matrix $\mathbf{M}^{-1}\mathbf{K}$. In order that the LNMs be the principal axes of the ellipsoid $\mathbf{q}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{K}\mathbf{q} = 1$, the matrix $\mathbf{M}^{-1}\mathbf{K}$ must be real, positive definite and symmetric [16]. This is the case if the mass matrix is proportional to identity, i.e., $\mathbf{M} = \alpha\mathbf{I}$. Accordingly, the LNMs are the principal axes of the ellipsoid $(1/\alpha)\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q} = 1$. This latter expression is to a constant, the expression of the potential energy in the $(q_1, \ldots, q_n)$ space. Since it is assumed that the mass matrix is proportional to identity, this is also the expression, to a constant, of the potential energy in the $(\xi_1, \ldots, \xi_n)$ space. This is another way to demonstrate that the LNMs are the principal axes of the bounding ellipsoid in the $(\xi_1, \ldots, \xi_n)$ space.